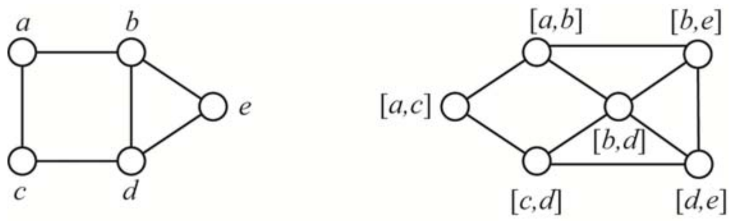
Line Graphs

Complement to Chapter 4, "The Case of the Hidden Inheritance"

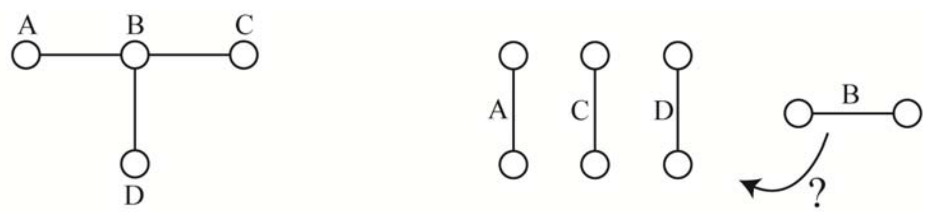
Starting with a graph G, we can associate a new graph with it, graph H, which we can also note as L(G) and which we call the *line graph* of G. This kind of graph is obtained by creating a vertex per edge in G and linking two vertices in H=L(G) if, and only if, the corresponding edges in G have an end in common.

In the example below, graph G contains six edges, which means that L(G) contains six vertices. The vertices [*a*,*b*] and [*a*,*c*] are linked by an edge in L(G) because the corresponding edges in G have the *a* vertex in common. However, there is no edge linking the vertices [*a*,*c*] and [*b*,*e*] in L(G) because those two edges in G have no ends in common.



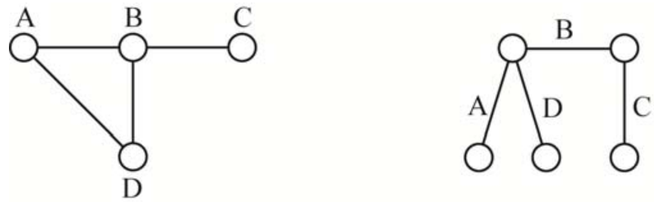
Graph G Its line graph L(G)

Graphs exist that are not line graphs. For example, the graph H below is not a line graph because if it were, there would have to exist a graph G such as H=L(G) and we would have to have three edges, A, C and D, in G with no common ends, and a fourth edge, B, in G with one end in common with the A, C and D edges, which is of course impossible, because any one edge only has two ends.



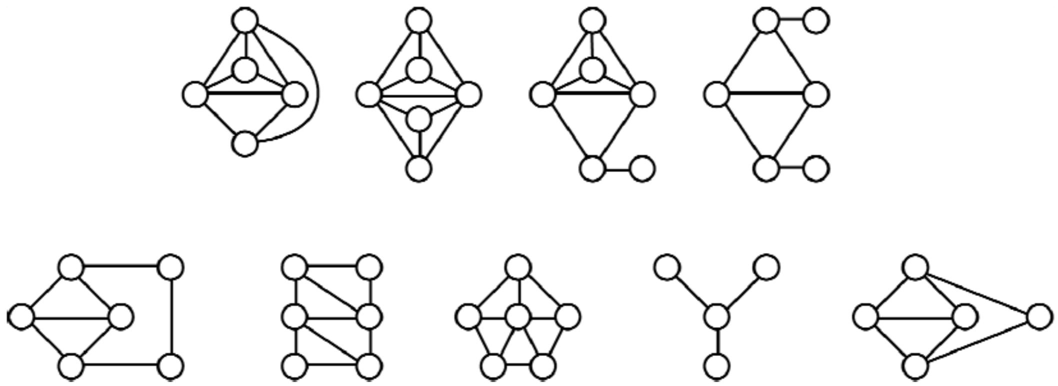
An H graph that is not a line graph The graph G for H=L(G)

Note that if the H graph above had an additional edge, for instance an edge linking A with D, then H would be a line graph since the B edge could then have an end in common with A and D and the other end in common with C.

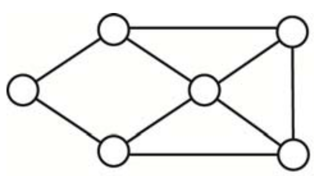


An H line graph The graph G for H=L(G)

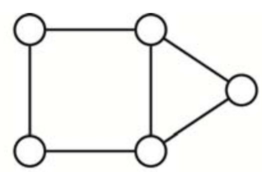
There exist, in fact, exactly nine configurations that do not correspond to a line graph. They are shown below. If an H graph contains at least one of these nine configurations, as is (meaning with no additional edges), then it is not a line graph. The second-last of these configurations is the one given as an example above.



Also, if an H graph contains none of these nine configurations, then there is a sole G graph for H=L(G), as long as no connected component of G is a triangle. The left-hand graph given at the beginning of this document is the only G graph whose right-hand graph is the line graph.

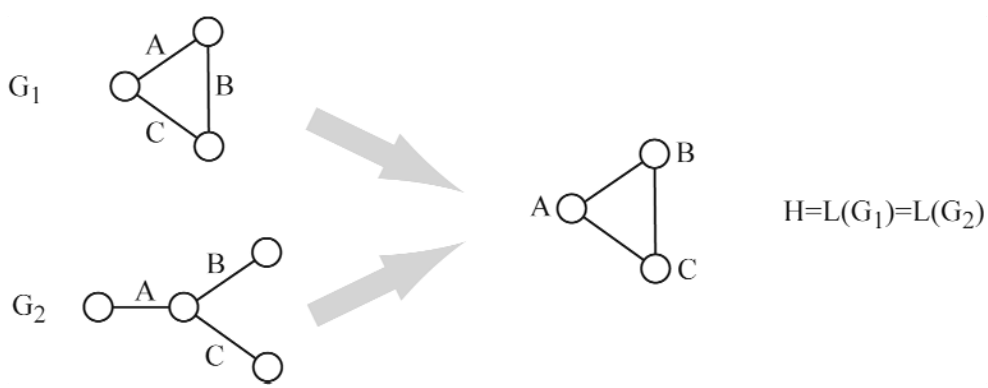


An H line graph

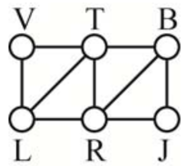


The sole G graph for H=L(G)

The exception mentioned above for G graphs containing connected components that are triangles comes from the fact that there are two graphs, G_1 and G_2, such that $L(G_1)$ and $L(G_2)$ are triangles.



To unmask the impostor in “The Case of the Hidden Inheritance,” Bonvin builds a graph representing the full treasure hunt, including the information contained in the part of the document supposedly found by the person demanding ransom. To do this, he creates a graph containing one vertex per path, and he links two paths when they have an end in common. The graph he built must, then, be the line graph for the graph in which the vertices are the intersections at the ends of the paths, and the edges are the paths themselves. Bonvin shows Manori the following graph, and Manori quickly realizes that it’s the sixth prohibited configuration.



In the part of the document which the person demanding ransom claims to have found, it seems that the violet and begonia paths (that is, vertices V and B) have no ends in common. That’s the only information that can be false, because there is no doubt as to the validity of the rest of the document. If we suppose instead that the violet and begonia paths do have an end in common, then we must add an edge between V and B, and this time we get a line graph, as Manori demonstrates.

The H line graph corresponding to the treasure hunt

The G graph for H=L(G)

Line graphs can be useful in terms of providing a different picture of a situation. Consider, for example, a problem with residential garbage collection. More specifically, let’s suppose that a transport company is tasked with collecting garbage in a Montréal neighbourhood. The company has been given a neighbourhood map showing all the streets to serve. The company only wants to use a single truck for all the collection. Also, to minimize the number of kilometres driven, the company would like to know whether there exists a route that travels exactly once along each street in the neighbourhood. The truck would then waste not a single kilometre. In graph theory terms, the company would like to know whether there is a *Eulerian* cycle in the graph.

Definition

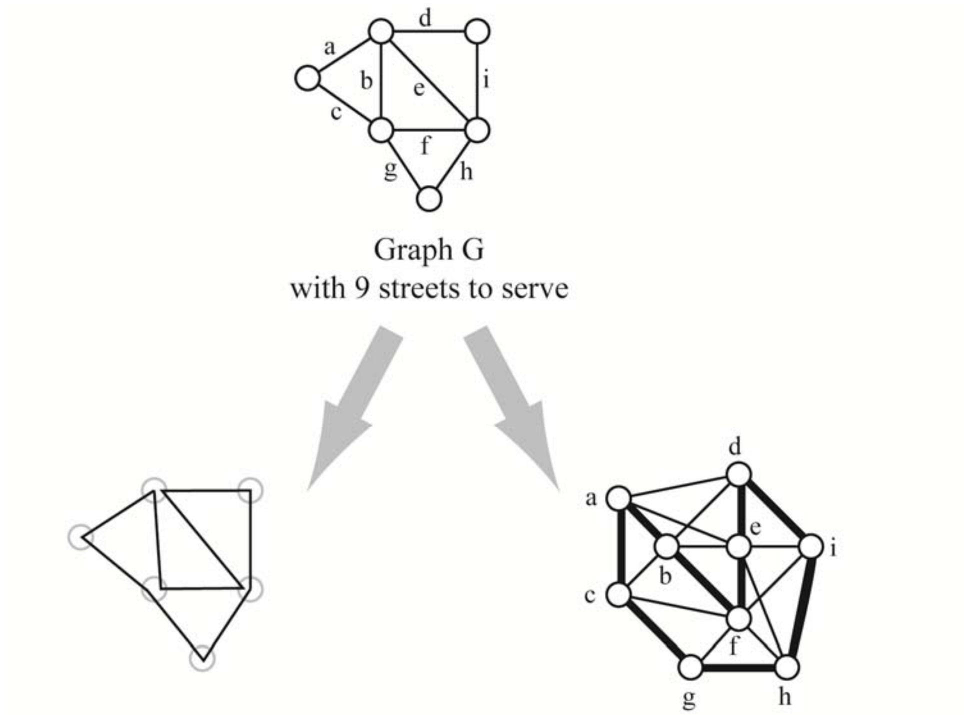
A cycle that travels exactly once over each edge of a graph is called “Eulerian.”

If we consider the line graph L(G) for G, we are led to ask whether there exists a route that travels only once by each vertex in L(G). Such a cycle is called *Hamiltonian*.

Definition

A cycle that travels exactly once over each vertex in a graph is called “Hamiltonian.”

In other words, to determine whether there is a Eulerian cycle in graph G, we can construct the line graph L(G) and determine if there is a Hamiltonian cycle in L(G). In the example below, we suppose that the company must collect garbage in nine streets, from a through i. The bottom left represents a Eulerian cycle in G, while the right-hand graph represents that same route, but in the form of a Hamiltonian cycle (in bold) in L(G).



Graph G
with 9 streets to serve

Route that travels exactly once
over each edge in G
(Eulerian cycle)

Route (in bold) that travels
exactly once over each vertex in L(G)
(Hamiltonian cycle)