

Planar Graphs

Complement to Chapter 2, "The Villas of the Bellevue"

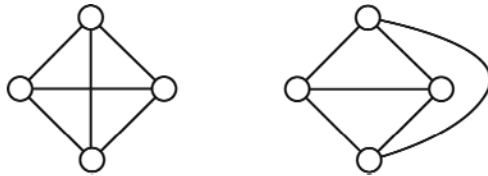
In the chapter "The Villas of the Bellevue," Manori gives Courtel the following definition.

Definition

"A graph is 'planar' if it can be drawn on a sheet of paper without any crossovers." Such a representation is called a "topological planar graph."

For example, the left-hand graph below is planar because by changing the way one edge is drawn, I can obtain the right-hand graph, which is in fact a different representation of the same graph, but without any edges crossing.

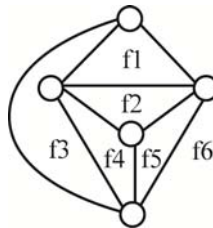
According to the definition above, these two graphs are planar, but only the one on the right is topological and planar.



Definition

In a topological planar graph, the zones delimited by the edges surrounding them are called "faces."

For example, in the graph representing the Courtels' villa, there are six faces named f1 to f6. Note that the outside contour of a topological planar graph is also a face. In our example, it is face f6.



V is the number of vertices in a topological planar graph, E is its number of edges and F is its number of faces. Euler demonstrated the following property.

Property 1

In any connected topological planar graph, we have

$$F = E - V + 2$$

This property allows us to demonstrate many results. For example, by noting S as the smallest number of edges delimiting a face, we get the following result.

Property 2

Any connected topological planar graph with V vertices and whose smallest face features S edges contains at most $\frac{S}{S-2}(V-2)$ edges.

Proof

Let's say G is a connected topological planar graph with V vertices, E edges and F faces and for which the contour of each face is made up of at least S edges. According to Euler's property, we get $F = E - V + 2$.

By adding up the number of edges around each face, we obtain exactly double the number of edges. In fact, each edge is counted twice, because it appears around the contour of exactly two faces.

Since the contour of each face contains at least S edges, we get $2E \geq SF$.

By using Euler's property, this inequality can be rewritten as follows:

$$2E \geq S(E-V+2)$$

Meaning that, if we group the E s to the right:

$$\frac{S}{S-2}(V-2) \geq E \quad \square$$

Thanks to this property, as Manori did to prove Courtel's typically Marseillais exaggeration, we can show that a connected planar graph with V vertices cannot contain more than $3V-6$ edges.

Property 3

Any connected planar graph with V vertices contains at most $3(V-2)$ edges.

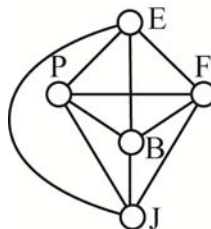
Proof

Let's say G is a connected planar graph with V vertices and E edges. Let's consider a representation R of G in which the edges do not cross one another. R is then topological and planar and has the same number of vertices and edges as G . Let S be the smallest number of edges making up the contour of a face in R .

Since each face contains at least three edges around its contour, we have $S \geq 3$, and

Property 2 shows us that $3(V-2) \geq \frac{S}{S-2}(V-2) \geq E$. □

As such, the graph below, which matches the description Courtel provided of his villa, cannot be planar because $3(V-2)$ gives 9, while E equals 10.



Now let's suppose that no face of a planar graph is triangular. In other words, if we choose three given vertices in the graph, there is always at least one edge missing between two of the three vertices. This gives us $S \geq 4$ and the upper bound on the number of edges becomes even smaller.

Property 4

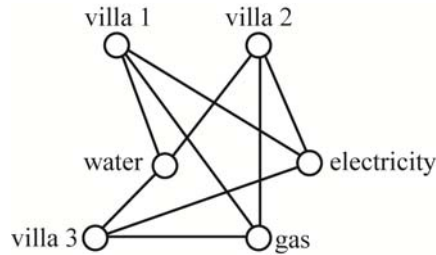
Any connected planar graph with V vertices and no triangles contains at most $2(V-2)$ edges.

Proof

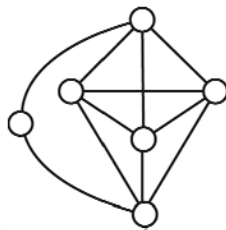
The proof is similar to the last one. In a connected planar graph G with V vertices, E edges and no triangles, we can consider a topological planar representation R of G in which the edges do not cross over one another. The R graph has the same number of vertices and edges as G and does not include any triangles. The smallest number of edges, S , making up the contour of a face in R is thus at least 4.

Property 2 shows us that $2(V-2) \geq \frac{S}{S-2}(V-2) \geq E$. □

By using this property, Manori manages to show that the graph below, representing the connections of each villa to the water, gas and electricity lines, is not planar. The graph includes no triangles and we get $2(V-2)=8$ with $E=9$.



We have observed that it is sometimes easy to show that a graph is not planar, but this is not always the case. In the example below, we have $V=6$ vertices, $E=11$ edges and the graph contains triangles. So we know that if this graph is planar, then $3(V-2) \geq E$. In our case, $3(V-2) = 12 > E$. This graph could, theoretically, be planar, but it isn't, because it is impossible to draw it while avoiding all crossovers between edges.



Property 2 helps us demonstrate a number of other results. For example, we can show that all planar graphs contain at least one vertex with a degree of 5 or higher.

Property 5

All planar graphs contain at least one vertex with a degree of 5 or higher.

Proof

Let G be a planar graph, and let G' be a connected component of G (note that $G'=G$ if G is connected). If we note as V and E the numbers of vertices and edges in G' , we know from Property 3 that $3(V-2) \geq E$, which can be rewritten:

$$3V \geq E+6 \tag{*}$$

Let's do some reasoning by contradiction. Let's suppose that all the vertices in G (and so also in G') have a degree of at least 6, and we'll see that we necessarily get something absurd: we are supposing that the total of the degrees of the vertices in G' is worth at least $6V$.

We have already seen that the total of the degrees of the vertices in a graph is equal to twice the number of edges. So we get $2E \geq 6V$, meaning

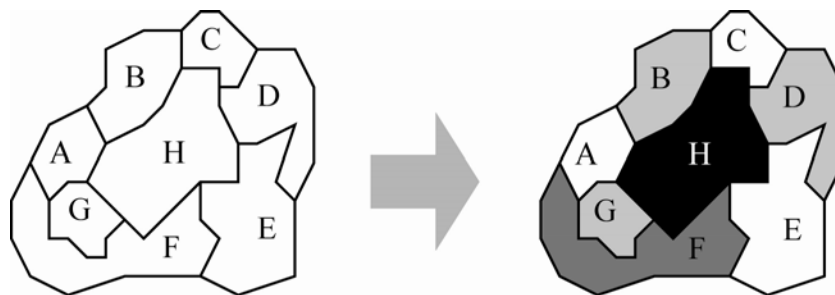
$$E \geq 3V \tag{**}$$

By grouping together the two inequalities (*) and (**) we then get $E \geq E+6$, which is, of course, impossible. □

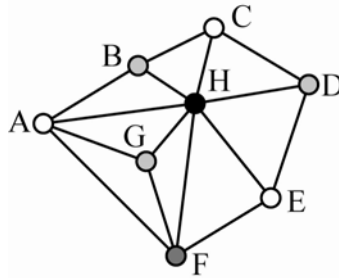
Let's look once again at the graph from the last chapter, in which we have one vertex per COPS participant and one edge between two participants when they have worked together in the past. If the COPS organizers end up defining their working groups in such a way that each participant has worked with exactly six other people in the group, then we know that the graph associated with a group is not planar, because all the vertices have a degree of 6.

Planar graphs were popularized by an Englishman, Francis Guthrie, who, in 1852, wondered if it were possible to colour all geographical maps using only four colours, with no two bordering countries being of the same colour. By representing each country with a vertex and linking two vertices with an edge if, and only if, the two countries represented by the vertices have a common border, we get a planar graph. Francis Guthrie wanted to know if it were always possible to colour the vertices of a planar graph using only four colours, with the sole constraint being that the ends of each edge must be of different colours (which is equivalent to requiring that countries with a common border be of different colours).

As an illustration, here is a four-colour colouring of an imaginary map.



The following graph represents this colouring.



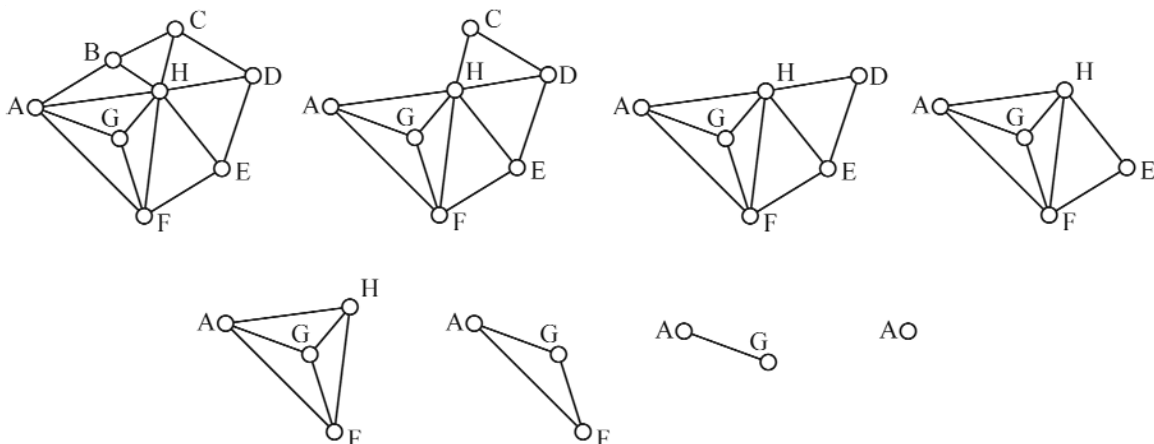
It's easy to colour any planar graph G using at most six colours (instead of four), with the sole constraint being that the ends of each edge must be of different colours. To do this, we can first order the vertices as follows (where V represents the number of vertices in graph G).

- Set G' equal to G and start a c counter at 0
- As long as G' is not empty, do as follows:
 - Choose the smallest-degree vertex in G' and place it in position $V-c$;
 - Remove this vertex from G' and move the counter up by one unit

As an illustration, for the graph below, we can choose B, C, D, E or G for the first step since they are all of a minimum degree of 3. Let's choose B, which is now put in 8th position.

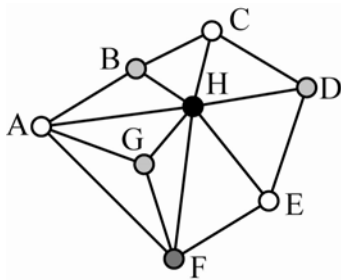
In the new graph obtained by deleting B, only C has a minimum degree of 2. So we place it in 7th position. Next it's D's turn to be placed in 6th position, then E in 5th. There only remain four vertices with a degree of 3. We can, for instance, choose to place vertex H in 4th position, then F in 3rd position, G in 2nd and lastly A in 1st. This gives us a list containing A, G, F, H, E, D, C and B, in that order.

The process is illustrated below.



We can then colour the vertices of G using the order set out in the list. By construction, every time we colour a vertex, it is the one with the lowest degree in the graph made up of the vertices that have already been coloured. Given that G is planar, we know because of Property 5 that its degree is necessarily below 6. In other words, when we colour a vertex, it is linked with at most five vertices that have already been coloured. So one of the six colours is always available.

For example, if we decide to always use the colour white when that is possible, light grey as a second choice, dark grey as a third choice, and black as a fourth, then when colouring vertices in the order A, G, F, H, E, D, C and B , we start by assigning white to A ; then G will be light grey, F dark grey, and H black. When it comes to vertex E , it can be given white or light grey, so we give it white as a first choice. We continue in this way until we get the colouring already shown above.



It is a bit difficult (but not excessively so) to colour a planar graph with a maximum of five colours. It was only in 1976 that we learned, thanks to Kenneth Appel and Wolfgang Haken, that four colours are indeed always enough. But that is very complicated to carry out in practice.